Math 10A
Final; Thursday, 8/9/2018
Time: 2:10 PM **Name:** ____________________
Instructor: Roy Zhao **Student ID:** ____________________

- **DO NOT OPEN THE FINAL UNTIL TOLD TO DO SO!**
- Do all problems as best as you can. The exam is 110 minutes long. You may not leave during the last 30 minutes of the exam.
- Use the provided sheets to write your solutions. You may use the back of each page for the remainder of your solutions; in such a case, put an arrow at the bottom of the page and indicate that the solution continues on the back page. **No extra sheets of paper can be submitted with this exam!**
- The exam is closed notes and book, which means: **no class notes, no review notes, no textbooks, and no other materials can be used during the exam.** You can only use your cheat sheet. The cheat sheet is two sides of one regular 8×11 sheet, handwritten.
- **NO CALCULATORS ARE ALLOWED DURING THE EXAM!**
- Justify all your answers, include all intermediate steps and calculations, and box your answers.

1. (10 points) Find the following limits.

   (a) (2 points) $\lim_{x\to 1} \frac{x^2+1}{x+2} =$

   Solution: Plug in $x = 1$ to get $\frac{1^2+1}{1+2} = \frac{2}{3}$.

   (b) (3 points) $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} =$

   Solution: We plug in 0 to get 0/0 so we use L'Hopital's and take the derivative of the top and bottom to get

   $$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x\to 0} \frac{e^x - 1}{2x}.$$

   Again we plug in $x = 0$ to get 0/0 so use use L'Hopital's again to get

   $$\lim_{x\to 0} \frac{e^x - 1}{2x} = \lim_{x\to 0} \frac{e^x}{2} = \frac{e^0}{2} = \frac{1}{2}.$$

   (c) (5 points) $\lim_{x\to\infty} \sqrt{x^2+x} - x =$

   Solution: We multiply by the conjugate $\sqrt{x^2+x} + x$ to get

   $$\lim_{x\to\infty} \sqrt{x^2+x} - x = \lim_{x\to\infty} \frac{x^2+x-x^2}{\sqrt{x^2+x} - x} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+x} + x}.$$

   Dividing the top and bottom by x, we get

   $$= \lim_{x\to\infty} \frac{1}{\sqrt{1+1/x} + 1} = \frac{1}{1+1} = \frac{1}{2}.$$

2. (15 points) For each part, find $\dfrac{dy}{dx}$.

(a) (5 points) $y = e^{\sin(x^2)}$.

Solution: $y' = e^{\sin(x^2)} \cdot \cos(x^2) \cdot 2x$.

(b) (5 points) $x^2 + y^2 = xy$. (You can leave your answer in terms of x and y)

Solution: Taking the derivative, we get $2x+2yy' = xy'+y$ so $y'(2y-x) = y-2x$ or $y' = \dfrac{y-2x}{2y-x}$.

(c) (5 points) $y = \displaystyle\int_1^{\sqrt{x}} \frac{t^2}{1+t^2} dt$.

Solution: Taking the derivative and using FTC, the answer is

$$\frac{(\sqrt{x})^2}{1+\sqrt{x}^2} \cdot \frac{1}{2\sqrt{x}} = \frac{x}{1+x}\frac{1}{2\sqrt{x}} = \frac{\sqrt{x}}{2+2x}.$$

3. (10 points) Find the following integrals.

(a) (5 points) $\int 2x^3\sqrt{x^2-1}dx =$

Solution: Let $u = x^2 - 1$. Then $du = 2xdx$ and hence

$$\int 2x^3\sqrt{x^2-1}dx = \int x^2\sqrt{u}du = \int (u+1)\sqrt{u}du = \int u^{3/2} + u^{1/2}du$$

$$= \frac{2u^{5/2}}{5} + \frac{2u^{3/2}}{3} + C = \frac{2(x^2-1)^{5/2}}{5} + \frac{2(x^2-1)^{3/2}}{3} + C.$$

(b) (5 points) $\int 4x^3 \arctan(x^2)dx =$

Solution: Let $u = x^2$ so $du = 2xdx$ and

$$\int 4x^3 \arctan(x^2)dx = \int 2x^2 \arctan(u)du = \int 2u \arctan(u)du.$$

We use integration by parts with $r = \arctan(u)$ and $dt = 2udu$ so $dr = \frac{1}{1+u^2}$ and $t = u^2$. This gives

$$= u^2 \arctan(u) - \int \frac{u^2}{1+u^2}du.$$

We use long division on the second integral to get $\frac{u^2}{1+u^2} = 1 - \frac{1}{1+u^2}$. This gives

$$= u^2 \arctan(u) - \int 1 - \frac{1}{1+u^2}du = u^2 \arctan(u) - u + \arctan(u) + C$$

$$= x^4 \arctan(x^2) - x^2 + \arctan(x^2) + C.$$

4. (5 points) Does $\int_2^\infty \frac{1}{\sqrt{x^2-1}}dx$ converge?

Solution: We want to compare it to $\frac{1}{x^p}$ for some power p. The thing we choose for p is $p = 1$ because $\sqrt{x^2-1} \approx \sqrt{x^2} = x$. We have $x^2 - 1 \leq x^2$ so $\sqrt{x^2-1} \leq \sqrt{x^2} = x$. Therefore

$$\int_2^\infty \frac{1}{\sqrt{x^2-1}}dx \geq \int_2^\infty \frac{1}{x}dx = \infty.$$

Therefore, the integral diverges.

5. (10 points) Suppose that I am currently standing 5 meters east of a bus. The bus starts moving north at a rate of 6 m/s. How fast is the bus moving away from me two seconds later?

Solution: Let x be how far north the bus has moved. We know that $\frac{dx}{dt} = 6$ and the distance between me and the bus is given by $z = \sqrt{5^2 + x^2}$. Taking the derivative of both sides and plugging in the fact that two seconds later $x = 12$, we get that

$$\frac{dz}{dt} = \frac{2xx'}{2\sqrt{25 + x^2}} = \frac{2 \cdot 12 \cdot 6}{2 \cdot \sqrt{25 + 144}} = \frac{12 \cdot 6}{13} = \frac{96}{13}.$$

6. (10 points) Find a second order recurrence relation or differential equation that has the following solutions.

(a) (5 points) $a_n = 2^n - 1$.

Solution: We write $1 = 1^n$ and so the base of the exponents are 2 and 1 which means the characteristic polynomial is $(\lambda - 2)(\lambda - 1) = \lambda^2 - 3\lambda + 2 = 0$. So, the recurrence relation is $a_n - 3a_{n-1} + 2a_{n-2} = 0$ or $a_n = 3a_{n-1} - 2a_{n-2}$.

(b) (5 points) $y = e^{2t}\sin(t)$.

Solution: The roots of the characteristic equation are $2 \pm 1i$ so the characteristic polynomial is $(\lambda - (2+i))(\lambda - (2-i)) = \lambda^2 - 4\lambda + 5$. Therefore, the differential equation is $y'' - 4y' + 5y = 0$.

7. (10 points) Solve the following IVPs.

(a) (5 points) $xy' = 2y + 2x^4, y(1) = 2$.

Solution: It is a first order differential equation that is linear so we want to use integration by parts. get it in the form $y' + P(x)y = Q(x)$ so it looks like $y' + (-2/x)y = 2x^3$. Now our integrating factor is $I(x) = e^{\int -2/x dx} = e^{-2\ln x} = x^{-2}$. Multiplying by our integrating factor gives

$$\frac{y'}{x^2} - \frac{2y}{x^3} = 2x.$$

We integrate to get $I(x)y$ on the right side or $y/x^2 = x^2 + C$ and $y = x^4 + Cx^2$. Now we plug in our initial condition that $y(1) = 2 = 1^4 + C1^2 = 1 + C$ so $C = 1$. Therefore, the solution is $y = x^4 + x^2$.

(b) (5 points) $e^{t^2}y' = ty^2, y(0) = 1$.

Solution: This is first order and separable so we separate to get

$$\frac{dy}{y^2} = \frac{t}{e^{t^2}}dt = te^{-t^2}dt.$$

Integrating both sides and using u substitution on the right with $u = -t^2$ and $du = -2tdt$ gives

$$\frac{-1}{y} = \frac{1}{-2}e^{-t^2} + C.$$

So $y = \frac{1}{e^{-t^2}/2 + C}$.. Now plug in $y(0) = 1$ to get $1 = \frac{1}{e^0/2 + C}$ so $1/2 + C = 1$ and $C = 1/2$. Therefore,

$$y = \frac{1}{e^{-t^2}/2 + 1/2} = \frac{2}{e^{-t^2} + 1}.$$

8. (10 points) Let $A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 5 & 1 \\ -4 & -8 & 7 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix}$.

(a) (2 points) Find $A\vec{v}$.

Solution:

$$A\vec{v} = \begin{pmatrix} 1 \cdot -2 + 2 \cdot 2 + -3 \cdot 3 \\ 2 \cdot -2 + 5 \cdot 2 + 1 \cdot 3 \\ -4 \cdot -2 + -8 \cdot 2 + 7 \cdot 3 \end{pmatrix} = \begin{pmatrix} -7 \\ 9 \\ 13 \end{pmatrix}.$$

(b) (8 points) Use Gaussian elimination to find the solution to $A\vec{x} = \vec{v}$.

Solution:

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 2 & 5 & 1 & 2 \\ -4 & -8 & 7 & 3 \end{array}\right) \xrightarrow{II-2I, III+4I} \left(\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & 1 & 7 & 6 \\ 0 & 0 & -5 & -5 \end{array}\right)$$

$$\xrightarrow{I-2II} \left(\begin{array}{ccc|c} 1 & 0 & -17 & -14 \\ 0 & 1 & 7 & 6 \\ 0 & 0 & -5 & -5 \end{array}\right) \xrightarrow{III/-5} \left(\begin{array}{ccc|c} 1 & 0 & -17 & -14 \\ 0 & 1 & 7 & 6 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{I+17III, II-7III} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right).$$

Therefore, the answer is $\vec{x} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$.

9. (10 points) Find the general solution to the system of differential equations

$$\begin{cases} y_1'(t) = 3y_2(t) \\ y_2'(t) = y_1(t) - 2y_2(t) \end{cases}.$$

Solution: We write this as $\vec{y}' = A\vec{y}$ with $A = \begin{pmatrix} 0 & 3 \\ 1 & -2 \end{pmatrix}$. The eigenvalues are when $(-\lambda)(-2-\lambda) - 3 = \lambda^2 + 2\lambda - 3 = (\lambda+3)(\lambda-1) = 0$, or when $\lambda = 1, -3$. For $\lambda = 1$, the eigenvector is gotten by looking at $A - \lambda I = \begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix}$ so it is $\vec{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. For $\lambda = -3$, we look at $A - \lambda I = \begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix}$ so $\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. The general solution is then

$$\vec{y} = c_1 e^t \vec{v}_1 + c_2 e^{-3t} \vec{v}_2 = \begin{pmatrix} 3c_1 e^t \\ c_1 e^t \end{pmatrix} + \begin{pmatrix} -c_2 e^{-3t} \\ c_2 e^{-3t} \end{pmatrix} = \begin{pmatrix} 3c_1 e^t - c_2 e^{-3t} \\ c_1 e^t + c_2 e^{-3t} \end{pmatrix}.$$

10. (10 points) Bubble True or False. (1 point for correct answer, 0 if incorrect)

(a) Ⓣ ● If $f(x)$ is not defined at $x = 2$, then $\lim_{x \to 2} f(x)$ doesn't exist.

Solution: The limit could exist even if the function does not (think $(x^2 - 9)/(x - 3)$ as $x \to 3$.

(b) Ⓣ ● The graph of $f(x-1)$ is the graph of $f(x)$ shifted 1 unit to the left.

Solution: It is shifted to the right.

(c) ● Ⓕ Using Simpson's method will give the exact answer when integrating $\int_0^1 x^3 + 3x^2 + 1dx$ with $n = 2$.

Solution: For a cubic equation, $K_4 = 0$ so there is no error.

(d) ● Ⓕ Changing the initial conditions for a linear homogeneous recurrence relation does not affect the bases of the exponential functions that appear in the formula for the solution.

(e) Ⓣ ● BVPs for second order linear homogeneous DEs with constant coefficients have either no solutions or infinitely many solutions.

Solution: It could also have one solution.

(f) ● Ⓕ The slope field of $\frac{dy}{dt} = \sin(t)$ will be the same if we shift it up or down.

Solution: Since the differential equation does not depend on y, the slope is the same regardless what t is and hence is the same if we shift it up or down.

(g) ● Ⓕ If we find two distinct solutions to $A\vec{x} = \vec{b}$, then $|A| = 0$.

Solution: The number of solutions is $0, 1, \infty$. Since there are at least two solutions, there are not 0 or 1 so there must be ∞ solutions so $|A| = 0$.

(h) ● Ⓕ If the augmented matrix $(A|\vec{b})$ is reduced into $(I|\vec{c})$ for some vector $\vec{c}$ by Gaussian elimination, then $A\vec{c} = \vec{b}$.

Solution: We use the augmented matrix to find the solution to $A\vec{x} = \vec{b}$ and so reducing it into $(I|\vec{c})$ means that $\vec{x} = \vec{c}$.

(i) Ⓣ ● An eigenvector can be the zero vector.

Solution: An eigenvector must always be nonzero.

(j) ● Ⓕ If 2 is an eigenvalue for A, then 4 is an eigenvalue for A^2.

Solution: If 2 is an eigenvalue for A, then $A\vec{v} = 2\vec{v}$ and $A^2\vec{v} = A(2\vec{v}) = 2^2\vec{v} = 4\vec{v}$ so 4 is an eigenvalue for A^2.